

On the subdivision of small categories

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Abstract

We present an intrinsic and concrete development of the subdivision of small categories, give some simple examples and derive its fundamental properties. As an application, we deduce an alternative way to compare the homotopy categories of spaces and small categories, by using partially ordered sets. This yields a new conceptual proof to the well-known fact that these two homotopy categories are equivalent.

Introduction

We present the subdivision of categories from the homotopy point of view, and illustrate this with some simple examples. This subdivision is not new, it has already appeared in some works by Anderson, Dwyer and Kan [1, 3]. Here, we derive the basic properties of the subdivision functor $C \mapsto Sd(C)$ from classical results on homotopy of categories, such as the famous Quillen's Theorem A. This way we obtain an intrinsic and geometric-style development of the theory.

Among the fundamental properties of the subdivision of categories, we emphasize theorems 21 and 32. The first one asserts that any category becomes a poset after applying the functor Sd twice, and the second relates the classifying spaces of a category and its subdivision by a homotopy equivalence. These results suggest that the homotopy type of the classifying space of any small category can be modelled by a poset, and therefore that the homotopy categories of small categories and posets are equivalent. This is proved in theorem 41.

Finally, we use some results of McCord [10] to relate the homotopy categories of posets and topological spaces. Combining these two equivalences we obtain the equivalence of categories

$$Ho(Top) \cong Ho(Cat),$$

which might be thought as a *categorical description of topological spaces*. This and the *combinatorial description of topological spaces* [4] are related by Quillen's theorem which asserts that the nerve functor is an equivalence at the homotopy level [6].

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1 Preliminaries

1.1 Homotopy categories

If M is a category and W is a family of arrows in M , there exists (eventually expanding the base universe) a category $M[W^{-1}]$, called *localization of M by W* , and a functor $p : M \rightarrow M[W^{-1}]$, called *localization functor*, which makes invertible all the arrows in W and which is universal for this property.

The category $M[W^{-1}]$ has the same objects than M and its arrows can be expressed as classes of paths involving arrows of M and formal inverses of arrows of W [4]. By using this description of the localization category, it is easy to prove the following result (cf. [13]).

1 Lemma. *Let $p : M \rightarrow M[W^{-1}]$ be a categorical localization. Then p induces a bijection*

$$\text{Hom}(F, G) \xrightarrow{\sim} \text{Hom}(Fp, Gp)$$

for every pair of functors $F, G : M[W^{-1}] \rightarrow N$.

When M is a category endowed with homotopical notions (e.g. model categories, simplicial categories, categories with cylinders) and when W is the class of weak equivalences of M , the localization of M by W is usually called the *homotopy category* of M , and is written by $Ho(M)$. The paradigmatic example is that of topological spaces and weak homotopy equivalences. We recall its definition.

2 Definition. The homotopy category $Ho(Top)$ is the localization of Top , the category of topological spaces, by the family of weak equivalences. Thus, $Ho(Top) = Top[W^{-1}]$ with

$$W = \{f : X \rightarrow Y \mid f_* : \pi_n(X, x) \rightarrow \pi_n(Y, f(x)) \text{ is an isomorphism } \forall n \forall x\}.$$

1.2 Homotopical notions in Cat

The other example we are going to consider is that of small categories. The category of small categories Cat is endowed with homotopical notions that one can lift from Top by using the classifying space functor $B : Cat \rightarrow Top$ [14]. We briefly recall from [12, 14] some definitions and results concerning this functor.

The category Δ is that whose objects are the finite ordinals $[q] = \{0 < 1 < \dots < q\}$ and whose arrows are the order preserving maps. We use the following standard notation: for $i = 0, \dots, q$ let $s_i : [q+1] \rightarrow [q]$ be the surjection which takes twice the value i , and let $d_i : [q-1] \rightarrow [q]$ be the injection whose image does not contain the value i .

If C is a small category, its *nerve* NC is the simplicial set whose q -simplices are the chains

$$X = (X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_q)$$

of q composable arrows of C . Formally, a q -simplex X is a functor $[q] \rightarrow C$ where $[q]$ is viewed as a category in the canonical way. Faces and degeneracies of NC are given by

composing adjacent arrows (or deleting the first or the last arrow) and inserting identities, respectively.

The *classifying space* BC of a category C is the geometric realization of its nerve, namely $BC = |NC|$. A functor $f : B \rightarrow C$ in \mathcal{Cat} is said to be a *weak equivalence* if Bf is a homotopy equivalence in \mathcal{Top} , and a small category C is said to be *contractible* if BC is so.

There is a homeomorphism $B(C \times D) \cong BC \times BD$ when, for instance, NC or ND has only finite non-degenerate simplices. In particular, denoting $I = [1]$, one has that a functor $C \times I \rightarrow D$ induces a continuous map $BC \times BI \rightarrow BD$. Thus, it follows that a natural transformation $f \cong g$ induces a homotopy $Bf \cong Bg$. Some simple and useful applications of it are the following.

3 Lemma. *If a functor admits an adjoint, then it is a weak equivalence.*

Proof. It suffices to consider the homotopies arising from the unit and the counit of the adjunction. \square

4 Lemma. *If a category has initial or final object, then it is contractible.*

Proof. In these cases the functor $C \rightarrow *$ admits an adjoint, $*$ being the one-arrow category. \square

5 Lemma. *Let $i : A \rightarrow B$ a fully faithful inclusion. If there is a functor $r : B \rightarrow A$ and a natural transformation $id_B \Rightarrow ir$, then i is a weak equivalence.*

Proof. The natural transformation $id_B \Rightarrow ir$ gives rise to another one $i \Rightarrow iri$, and since i is fully faithful, this is the same than a natural transformation $id_A \Rightarrow ri$. The result now follows from the fact that a natural transformation induces a homotopy. \square

We complete this review by recalling the definition of $Ho(\mathcal{Cat})$.

6 Definition. The homotopy category $Ho(\mathcal{Cat})$ is the localization of \mathcal{Cat} by the family of weak equivalences, that is, $Ho(\mathcal{Cat}) = \mathcal{Cat}[W^{-1}]$ with

$$W = \{f : C \rightarrow D \mid Bf : BC \rightarrow BD \text{ is a homotopy equivalence}\}.$$

7 Remark. \mathcal{Cat} admits a different homotopy structure than the one used here (cf. [11]). The functors which become homotopy equivalences after taking the classifying space functor are sometimes called *topological weak equivalences* to avoid confusions.

1.3 Quillen's Theorem A

Quillen's Theorem A provides a criteria to recognize when a functor is a weak equivalence. We fix some notations and recall it from [12, §1].

If $f : C \rightarrow D$ is a functor and if T is an object of D , then the *fiber* $f^{-1}T$ of f over T is the subcategory of C whose objects and arrows are those which f carries into T and id_T , respectively. The *left fiber* f/T of f over T is the category of pairs (X, u) with X an object of C and $u : fX \rightarrow T$, where an arrow between pairs $(X, u) \rightarrow (X', u')$ is a map

$v : X \rightarrow X'$ in C such that $u'f(v) = u$. The *right fiber* Y/T is defined dually. By an abuse of notation, we shall write C_T , C/T and T/C for the fiber, left fiber and right fiber, respectively.

8 Theorem (Quillen's Theorem A). *The functor $f : C \rightarrow D$ is a weak equivalence if it satisfies either (i) C/T is contractible for every object T of D , or (ii) T/C is contractible for every object T of D .*

Given $f : C \rightarrow D$, an arrow $X \xrightarrow{u} Y$ of C is said to be *cocartesian* if every arrow $X \xrightarrow{v} Y'$ such that $f(u) = f(v)$ factors as $\tilde{v} \circ u$ with $f(\tilde{v}) = id_{f(Y)}$ in a unique way.

$$\begin{array}{ccc} & & Y' \\ & \nearrow \forall v & \uparrow \exists! \tilde{v} \\ X & \xrightarrow{u} & Y \\ f(X) & \xrightarrow{f(u)} & f(Y) \end{array}$$

The functor f is a *pre-cofibration* if for each arrow $S \xrightarrow{\phi} T$ of D and for each $X \in C_S$ there is a cocartesian arrow $X \xrightarrow{u} Y$ over ϕ . The functor f is a *cofibration* if it is a pre-cofibration and also cocartesian arrows are closed under compositions.

Cartesian arrows are defined dually, as well as *pre-fibrations* and *fibrations*.

When $f : C \rightarrow D$ is a pre-cofibration, the inclusion $C_T \rightarrow C/T$ admits a right adjoint, called *base-change*, that push-forward an object (X, ϕ) along a cocartesian arrow $X \rightarrow Y$ over ϕ . This remark and its dual, combined with lemma 3, yield the following corollary.

9 Corollary. *Let $f : C \rightarrow D$ be a functor which is either a pre-fibration or a pre-cofibration. If C_T is contractible for every object T of D , then f is a weak equivalence.*

2 Subdivision of categories

2.1 The construction of $Sd(C)$

Let C be a small category. With Δ/C we mean the left fiber over C of the embedding $\Delta \rightarrow Cat$. It has the simplices of NC as objects, and given X and Y simplices of dimensions q and p , a morphism $(Y, \xi, X) : X \rightarrow Y$ in Δ/C consists of an order preserving map $\xi : [q] \rightarrow [p]$ such that $Y \circ \xi = X$. We write ξ_* instead of (Y, ξ, X) when there is no place to confusion.

10 Remark. Note that if there is a map $X \rightarrow Y$ in Δ/C , then the sequence $X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_q$ is obtained from $Y_0 \rightarrow Y_1 \rightarrow \dots \rightarrow Y_p$ by composing some arrows and inserting some identities (X is a degeneration of a face of Y).

Let $X \in NC_q$, and let $s : [q+1] \rightarrow [q]$ be a surjection. If $d, d' : [q] \rightarrow [q+1]$ are the two right inverses of s , then we say that $d_*, d'_* : X \rightarrow Xs$ are *elementary equivalent*, and we write $d_* \approx d'_*$. Note that \approx is reflexive and symmetric. We define \sim as the minor equivalence relation on the arrows of Δ/C which is compatible with the composition and satisfies $\xi_* \approx \xi'_* \Rightarrow \xi \sim \xi'_*$. We say that ξ_* and ξ'_* are *equivalent* if $\xi_* \sim \xi'_*$. With $[\Delta/C]$ we denote the quotient category with the same objects than Δ/C and arrows the classes under \sim .

11 Definition. The *subdivision* of C , denoted $Sd(C)$, is the full subcategory of $[\Delta/C]$ whose objects are the non-degenerate simplices of NC .

We describe the situation with the following diagram,

$$\begin{array}{ccc} & \Delta/C & \\ & \downarrow & \\ Sd(C) & \longrightarrow & [\Delta/C] \end{array}$$

where $Sd(C) \rightarrow \Delta/C$ is just the inclusion and $\Delta/C \rightarrow [\Delta/C]$ is the functor which maps an object to itself and an arrow $\xi_* : X \rightarrow Y$ to its class $[\xi_*]$ under \sim .

12 Remark. Notice that this is not the subdivision given in [7, III-10.1] or [9, IX-5]. Indeed, our construction is equivalent to that of [1, §2], as it can be deduced from lemma 18. Our definition describes completely the arrows of the subdivision category as homotopy-like equivalences of maps, where a degeneration of a simplex plays the role of a cylinder of it.

13 Remark. This subdivision gives rise to a functor $Cat \rightarrow Cat$ which equals the composition $c \circ sd \circ N$, where sd denotes Kan's subdivision of simplicial sets [8]. However, we believe that the intrinsic definition that we present here might be of interest, as it clarifies some aspect of subdivision of categories.

14 Example. If C is the category $0 \xrightleftharpoons[b]{a} 1$, then the full subcategory of Δ/C generated by the non-degenerate objects is

$$\begin{array}{ccc} 0 & \xrightarrow{\quad} & a \\ & \searrow \nearrow & \\ 1 & \xrightarrow{\quad} & b \end{array}$$

where a and b denote the non-trivial arrows of C . This is also the subdivision $Sd(C)$, since no identification is possible. The classifying space $B(Sd(C))$ is the 1-sphere S^1 .

15 Example. If C is the two-object simply connected groupoid $0 \rightleftharpoons 1$, then NC has two non-degenerate simplices on each dimension q , say $0101\dots$ and $1010\dots$. If $q < p$, then there are several arrows in Δ/C between a q -simplex and a p -simplex, but is not hard to see that any two of them are equivalent. Hence, it follows that $Sd(C)$ is the poset

$$\begin{array}{ccccccc} 0 & \xrightarrow{\quad} & 01 & \xrightarrow{\quad} & 010 & \xrightarrow{\quad} & \dots \\ & \searrow \nearrow & & \searrow \nearrow & & \searrow \nearrow & \\ 1 & \xrightarrow{\quad} & 10 & \xrightarrow{\quad} & 101 & \xrightarrow{\quad} & \dots \end{array}$$

Notice that $Sd(C)$ is the colimit of its subcategories $Sd(C)_{\leq n}$ formed by the simplices of dimension $\leq n$. Since $B(Sd(C)_{\leq n}) = S^n$ and since B commutes with directed colimits, it follows that $B(Sd(C))$ is homeomorphic to the infinite dimensional sphere S^∞ .

2.2 Some fundamental properties

If X is an object of Δ/C , we denote by q_X its dimension as a simplex of NC .

16 Definition. A map $\xi_* : X \rightarrow Y$ in Δ/C is a *surjection* if $\xi : [q_X] \rightarrow [q_Y]$ is so.

Note that if there is a surjection $X \rightarrow Y$, then X is a degeneration of Y .

17 Lemma. A surjection $\xi_* : X \rightarrow Y$ in Δ/C induces an isomorphism $[\xi_*] : X \rightarrow Y$ in $[\Delta/C]$.

Proof. It is sufficient to consider the case $q_X = q_Y + 1$, for any surjection can be expressed as a composition of some of the s_i . Thus, suppose that $\xi = s_i : [q + 1] \rightarrow [q]$, where $q = q_Y$. Then $X = Ys_i$ and the maps $(d_{i+1})_*, (d_{i+2})_* : X \rightarrow Xs_{i+1}$ are elementary equivalent. From the simplicial identities it follows that

$$(s_i)_*(d_{i+1})_* = id : Y \rightarrow Y$$

and that

$$(d_{i+1})_*(s_i)_* = (s_i)_*(d_{i+2})_* \sim (s_i)_*(d_{i+1})_* = id : X \rightarrow X.$$

Hence, $(s_i)_* : X \rightarrow Y$ and $(d_{i+1})_* : Y \rightarrow X$ are inverses modulo equivalences. \square

18 Lemma. If a functor $\Delta/C \rightarrow D$ carries surjections into isomorphisms, then it factors as $\Delta/C \rightarrow [\Delta/C] \rightarrow D$ in a unique way. Thus, $[\Delta/C]$ is the localization of Δ/C by the surjections.

Proof. If it exists, the factorization is unique because $\Delta/C \rightarrow [\Delta/C]$ is surjective on objects and on arrows. Let $f : \Delta/C \rightarrow D$ be a functor which carries surjections into isomorphisms. If $d_* \approx d'_* : X \rightarrow Xs$ are elementary equivalent maps and $s_* : Xs \rightarrow X$ is their left inverse, then $f(d_*) = f(s_*)^{-1} = f(d'_*)$. Thus, the relation $\xi_* \sim_f \xi'_* \iff f(\xi_*) = f(\xi'_*)$ is compatible with the composition and satisfies $\xi_* \approx \xi'_* \Rightarrow \xi_* \sim_f \xi'_*$. Therefore, $\xi_* \sim \xi'_* \Rightarrow \xi_* \sim_f \xi'_*$ and f factors through $[\Delta/C]$. \square

Let $\dim : Sd(C) \rightarrow \mathbb{N}_0$ be the functor $X \mapsto q_X$ which assigns to each non-degenerate simplex X its dimension.

19 Lemma. If there is a non-identity arrow $X \rightarrow Y$ in $Sd(C)$, then $\dim(X) < \dim(Y)$.

Proof. Let $[i_*] : X \rightarrow Y$ be an arrow of $Sd(C)$, with $i : [q_X] \rightarrow [q_Y]$ an order preserving map. Then $X = Yi : [q_X] \rightarrow C$ and i must be injective because X is a non-degenerate simplex of NC . Therefore $q_X \leq q_Y$, and $q_X = q_Y$ if and only if $i = id_{[q_X]} = id_{[q_Y]}$. \square

20 Corollary. If $f : X \rightarrow Y$ is an isomorphism in $Sd(C)$, then $X = Y$ and $f = id_X$.

Following the terminology of [5, §5], we have proved that $\dim : Sd(C) \rightarrow \mathbb{N}_0$ is a *linear extension* of the subdivision category $Sd(C)$, and that the latter is a *direct category*. This is not true for Δ/C nor $[\Delta/C]$, and here lies one reason for our construction.

21 Theorem. $Sd^2(C)$ is a poset for every category C .

Proof. We must show that for every pair X, Y of objects of $Sd^2(C)$, (i) there is at most one arrow $X \rightarrow Y$, and (ii) the existence of arrows $X \rightarrow Y$ and $Y \rightarrow X$ implies $X = Y$.

Assertion (ii) is an immediate corollary of lemma 19, so let us prove (i).

An object X of $Sd^2(C)$ is a non-degenerate simplex of $N(Sd(C))$, that is, a chain of non-trivial composable arrows

$$X = (X^0 \rightarrow X^1 \rightarrow \dots \rightarrow X^{q_X})$$

of $Sd(C)$, where X^i is a non-degenerate simplex of NC for each i . Note that $\dim(X^i) < \dim(X^{i+1})$ by lemma 19.

Fix two non-degenerate simplices X and Y of $N(Sd(C))$. We will show that there exists at most one order map $\xi : [q_X] \rightarrow [q_Y]$ such that $X = Y\xi$, from where (i) follows. Suppose that ξ, ξ' are such that $X = Y\xi = Y\xi'$. As we have pointed out, $\dim(Y^j) < \dim(Y^{j+1})$ for all j , so $X^i = Y^{\xi(i)} = Y^{\xi'(i)}$ implies that $\xi(i) = \xi'(i)$ and therefore $\xi = \xi'$. \square

22 Remark. The same argument of above proves that $Sd(C)$ is a poset for every direct category C in the sense of [5].

2.3 Functoriality of the subdivision

If X is a simplex of NC , we might think of X as a sequence of composable arrows, say $X = (f_1, \dots, f_{q_X})$. Let $p_X = \#\{j \mid f_j \neq id\}$ be the number of non-identity arrows that appear in X , and let $r(X) = (f_{i_1}, \dots, f_{i_{p_X}})$ be the sequence obtained from X by deleting the identities. Then X is a degeneration of $r(X)$ – viewed as simplices of NC – and $r(X)$ is a non-degenerate p_X -simplex. Moreover, $X = r(X)\alpha_X$ with $\alpha_X : [q_X] \rightarrow [p_X]$ the surjective order map defined by $\alpha_X(i-1) = \alpha_X(i) \iff f_i = id$.

If $\xi_* : X \rightarrow Y$ is an arrow in Δ/C , we define $r(\xi_*)$ as the arrow $r(X) \rightarrow r(Y)$ in $Sd(C)$ given by the composition $[(\alpha_Y)_*][\xi_*][(\alpha_X)_*]^{-1}$ in $[\Delta/C]$.

$$\begin{array}{ccc} X & \xrightarrow{[\xi_*]} & Y \\ [(\alpha_X)_*] \downarrow & & \downarrow [(\alpha_Y)_*] \\ r(X) & \xrightarrow{r(\xi_*)} & r(Y) \end{array}$$

Note that $[(\alpha_X)_*]$ is invertible by lemma 17.

With these definitions $r = r_C : \Delta/C \rightarrow Sd(C)$ is a functor which maps surjection into identities, and by lemma 18 it induces a new one $[\Delta/C] \rightarrow Sd(C)$, also denoted by r_C .

Let $i_C : Sd(C) \rightarrow [\Delta/C]$ be the canonical inclusion. Clearly $r_C i_C = id$, and by lemma 17 we have that $\alpha : id \Rightarrow i_C r_C$, $X \mapsto [(\alpha_X)_*]$ is a natural isomorphism. Thus, $i_C : Sd(C) \rightarrow [\Delta/C]$ is an equivalence of categories with inverse $r_C : [\Delta/C] \rightarrow Sd(C)$. In particular, $Sd(C)$ is a skeleton of $[\Delta/C]$ as it follows from corollary 20.

23 Lemma. *The construction $C \mapsto Sd(C)$ is functorial.*

Proof. A functor $f : C \rightarrow D$ induces a new one $f_* : \Delta/C \rightarrow \Delta/D$ by mapping a simplex X to $f \circ X$. This functor clearly sends surjections into surjections. Then, it induces a

functor $[f_*] : [\Delta/C] \rightarrow [\Delta/D]$, which does not necessarily carry $Sd(C)$ into $Sd(D)$. Thus, we must define $Sd(f)$ as the composition $r_D[f_*]i_C$.

$$\begin{array}{ccc} Sd(C) & \xrightarrow{i_C} & [\Delta/C] \\ Sd(f) \downarrow & & \downarrow [f_*] \\ Sd(D) & \xleftarrow{r_D} & [\Delta/D] \end{array}$$

To prove that it is functorial we must verify that $Sd(id) = id$ and that $Sd(g)Sd(f) = Sd(gf)$. The first assertion follows because $Sd(id) = r_C[id_*]i_C = r_Ci_C = id_{Sd(C)}$. About the other, if $f : C \rightarrow D$ and $g : D \rightarrow E$ then the natural isomorphism $\alpha : id \cong i_{Dr_D}$ induces another one

$$Sd(gf) = r_E[(gf)_*]i_C = r_E[g_*][f_*]i_C \cong r_E[g_*]i_{Dr_D}[f_*]i_C = Sd(g)Sd(f)$$

of functors $Sd(C) \rightarrow Sd(E)$. It follows from corollary 20 that the natural isomorphism $Sd(gf) \cong Sd(g)Sd(f)$ must be the identity, and hence we have proved that $Sd(gf) = Sd(g)Sd(f)$ and that Sd is a functor indeed. \square

24 Remark. It follows from proposition 21 that Sd^2 lifts to a functor $l : Cat \rightarrow PoSet$.

$$\begin{array}{ccc} & PoSet & \\ & \downarrow j & \\ Cat & \xrightarrow{Sd^2} & Cat \end{array} \quad \begin{array}{c} \nearrow l \\ \searrow \end{array}$$

Here j denotes the canonical inclusion $PoSet \rightarrow Cat$. Next section we will show that l is a homotopy inverse for j .

2.4 Relationship between a category and its subdivision

Recall the functor $sup : \Delta/C \rightarrow C$ [6, §3]: Given an object $X : [q_X] \rightarrow C$ of Δ/C , $sup(X) = X_{q_X}$ is the last object of the sequence X . For an arrow $\xi_* : X \rightarrow Y$ in Δ/C , recall that $sup(\xi_*) : X_{q_X} \rightarrow Y_{q_Y}$ is the composition of the arrows of Y between $Y_{\xi(q_X)}$ and Y_{q_Y} , namely $sup(\xi_*) = Y(\xi(q_X) \rightarrow q_Y)$.

The functor sup maps surjections into identities, since a surjective map $[p] \rightarrow [q]$ preserves final element. It follows from lemma 18 that sup factors through the quotient and induces a functor $[\Delta/C] \rightarrow C$ which will be denoted by $[sup]$.

$$\begin{array}{ccc} & \Delta/C & \\ & \downarrow & \searrow sup \\ Sd(C) & \xrightarrow{i_C} & [\Delta/C] \xrightarrow{[sup]} C \end{array}$$

25 Definition. The functor $\varepsilon_C : Sd(C) \rightarrow C$ is defined as the composition $[sup] \circ i_C$ of the bottom of the diagram of above.

26 Lemma. *The functor ε_C is natural in C . It gives rise to a natural transformation, denoted $\varepsilon : Sd \Rightarrow id_{\mathcal{Cat}}$.*

Proof. Given a map $f : C \rightarrow D$ in \mathcal{Cat} , we have the following diagram.

$$\begin{array}{ccccc} Sd(C) & \xrightarrow{i_C} & [\Delta/C] & \xrightarrow{[sup]} & C \\ Sd(f) \downarrow & & [f_*] \downarrow & & f \downarrow \\ Sd(D) & \xrightarrow{i_D} & [\Delta/D] & \xrightarrow{[sup]} & D \end{array}$$

Clearly the map sup is natural, and it follows from this that $[sup]$ is also natural. Thus, the right square of above is commutative. The left square does not commute, but there is a natural isomorphism $[f_*]i_C \Rightarrow i_D Sd(f)$ which consists of a surjective map

$$\alpha_{fX} : fX \Rightarrow i_D r_D fX$$

for each object X of $Sd(C)$ (see the definition of α in the previous subsection). Finally, as $[sup]$ carries surjections into identities, the big square commutes and the lemma follows. \square

Next we shall prove that the functor $\varepsilon_C : Sd(C) \rightarrow C$ is a weak homotopy equivalence. To do that, we first study the left fibers of ε_C .

Fix some object T of C . Let (X, f) be an object of $(\Delta/C)/T$, namely the left fiber of sup over T . Thus, $f : S \rightarrow T$ is a map in C , and $X = (X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_{q_X-1} \rightarrow S)$ is an object of Δ/C whose top element is S . We define $r(X, f)$ as the object of Δ/C obtained by extending X with f .

$$r(X, f) = (X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_{q_X-1} \rightarrow S \xrightarrow{f} T)$$

The assignment $(X, f) \mapsto r(X, f)$ is functorial: given $(X, f) \xrightarrow{\xi_*} (Y, g)$, we define $r(\xi_*) : r(X, f) \rightarrow r(Y, g)$ as the map of Δ/C induced by the order map

$$[q_X + 1] \rightarrow [q_Y + 1] \quad j \mapsto \xi(j) \quad (0 \leq j \leq q_X), \quad q_X + 1 \mapsto q_Y + 1.$$

This way we have a functor $r : (\Delta/C)/T \rightarrow (\Delta/C)_T$ into the fiber, which is some kind of retraction for the fully faithful canonical map $i : (\Delta/C)_T \rightarrow (\Delta/C)/T$. Indeed, given (X, f) in $(\Delta/C)/T$, there is a natural map $d_* : X \rightarrow r(X, f)$ induced by the injection $d = d_{q_X+1} : [q_X] \rightarrow [q_X + 1]$. Clearly, $sup(d_*) = f$ and d_* is also a map in $(\Delta/C)/T$, hence we have a natural transformation $id \Rightarrow ir : (\Delta/C)/T \rightarrow (\Delta/C)/T$.

27 Lemma. *The inclusion $(\Delta/C)_T \rightarrow (\Delta/C)/T$ is a weak equivalence.*

Proof. Follows from lemma 5 and the paragraph of above. \square

28 Remark. Note that the map $d_* : X \rightarrow r(X, f)$ is not a cocartesian arrow. As an example, consider an arrow $f : X \rightarrow T$ in C , and let (X, f) be the correspondent zero-dimension object of $(\Delta/C)/T$. Then the two maps

$$[(d_1)_*], [(d_2)_*] : r(X, f) = (X \xrightarrow{f} T) \rightarrow (X \xrightarrow{f} T \xrightarrow{id} T)$$

are different ways to factor $(X) \rightarrow (X \xrightarrow{f} T \xrightarrow{id} T)$ through d_* .

Now we shall prove that the functor $r : (\Delta/C)/T \rightarrow (\Delta/C)_T$ gives rise to a new one $[r] : [\Delta/C]/T \rightarrow [\Delta/C]_T$ between the left fiber and the actual fiber of $[sup]$. In order to do that, we have to show that r carries equivalent maps into equivalent maps. To prove this, we will need a more explicit description of the relation \sim .

29 Remark. We say that $\xi_* \sim_1 \xi'_*$ if there are factorizations $\xi_* = \xi_*^1 \xi_*^2 \dots \xi_*^n$ and $\xi'_* = \xi_*'^1 \xi_*'^2 \dots \xi_*'^n$ such that $\xi_*^i \approx \xi_*'^i$ for each i . Note that \sim_1 is reflexive and symmetric. We call \sim_2 to the equivalence relation generated by \sim_1 . Thus, $\xi_* \sim_2 \xi'_*$ iff there is a sequence

$$\xi_* \sim_1 h_*^1 \sim_1 h_*^2 \sim_1 \dots \sim_1 h_*^N \sim_1 \xi'_*.$$

It is easy to see that $\xi_* \sim \xi'_* \iff \xi_* \sim_2 \xi'_*$, since \sim_2 is an equivalence relation which contains the elementary equivalences and is compatible with the composition (this gives \Rightarrow) and \sim is an equivalence relation which contains \sim_1 (this gives \Leftarrow).

30 Lemma. Let $\xi_*, \xi'_* : (X, f) \rightarrow (Y, g)$ be maps of $(\Delta/C)/T$ such that $\xi_* \sim \xi'_*$ viewed as maps of Δ/C . Then $r(\xi_*) \sim r(\xi'_*)$.

Proof. First of all, observe that if $\xi_* \approx \xi'_*$, then $r(\xi_*) \approx r(\xi'_*)$.

Secondly, if $\xi_* \sim_1 \xi'_*$ then there are factorizations $\xi_* = \xi_*^1 \xi_*^2 \dots \xi_*^n$ and $\xi'_* = \xi_*'^1 \xi_*'^2 \dots \xi_*'^n$ such that $\xi_*^i \approx \xi_*'^i$ for each i . A priori these are just maps in Δ/C , but since the target of ξ_* and ξ'_* is an object in $(\Delta/C)/T$, then we can think of these maps as arrows in the left fiber. By applying the functor r we obtain factorizations $r(\xi_*^1)r(\xi_*^2)\dots r(\xi_*^n)$ and $r(\xi_*'^1)r(\xi_*'^2)\dots r(\xi_*'^n)$ of $r(\xi_*)$ and $r(\xi'_*)$ which together with previous paragraph imply that $r(\xi_*) \sim_1 r(\xi'_*)$.

Finally, if $\xi_* \sim_2 \xi'_*$, then $r(\xi_*) \sim_2 r(\xi'_*)$ by an inductive argument.

The lemma follows from remark 29. □

31 Lemma. The inclusion $[\Delta/C]_T \rightarrow [\Delta/C]/T$ is a weak equivalence.

Proof. Follows from lemmas 5 and 30. □

The following theorem allow us to consider $Sd(C)$ as an algebraic model for the homotopy type of BC , locally simpler than C .

32 Theorem. The functor $\varepsilon_C : Sd(C) \rightarrow C$ is a weak equivalence for every C .

Proof. The functor ε_C factors as $[sup] \circ i_C$. Since i_C is an equivalence of categories, it is a weak equivalence (cf. lemma 3) and we just need to prove that $[sup]$ is a weak equivalence.

We will apply theorem 8, so we need to prove that the left fibers of $[sup]$ are contractible. By lemma 31 it is sufficient to prove that the fiber $[\Delta/C]_T$ is contractible for each object T of C .

Given T , we will prove that $[\Delta/C]_T$ has an initial object and the result will follow from lemma 4. This initial object is T , viewed as a 0-simplex of NC . If X is any object of $[\Delta/C]_T$, then the (q_X) -th inclusion $\alpha : [0] \rightarrow [q_X]$ induces a map $[\alpha_*] : T \rightarrow X$ in $[\Delta/C]_T$.

If $[\beta_*] : T \rightarrow X$ is any other map in $[\Delta/C]_T$, we have to prove that $\alpha_* \sim \beta_*$. Consider the order map $h : [1] \rightarrow [q_X]$ given by $h(0) = \beta(0)$ and $h(1) = q_X$.

$$\begin{array}{ccc}
 (T \xrightarrow{id} T) & \xrightarrow{h_*} & (X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_{q_X}) \\
 \begin{array}{c} (d_0)_* \uparrow \\ (s_0)_* \downarrow \\ (d_1)_* \uparrow \end{array} & \xrightarrow{\alpha_*} & \\
 (T) & \xrightarrow{\beta_*} &
 \end{array}
 \quad \begin{array}{l} h_*(d_0)_* = \alpha_* \\ h_*(d_1)_* = \beta_* \end{array}$$

Then $\alpha = hd_0$ and $\beta = hd_1$, and because $X(\beta(0) \rightarrow q_X) = \varepsilon_C([\beta_*]) = id_T$ it follows that $Xh = Ts_0$ is a degeneration of T , $(d_0)_* \approx (d_1)_*$ and therefore $\alpha_* \sim \beta_*$. \square

33 Corollary. *The functor $Sd : Cat \rightarrow Cat$ preserves weak equivalences.*

Proof. If $f : C \rightarrow D$ is a weak equivalence in Cat , it follows from theorem 32 and the square

$$\begin{array}{ccc}
 Sd(C) & \xrightarrow{\varepsilon_C} & C \\
 Sd(f) \downarrow & & \downarrow f \\
 Sd(D) & \xrightarrow{\varepsilon_D} & D
 \end{array}$$

that $Sd(f)$ is also a weak equivalence. \square

34 Remark. Given (X, f) an object of $(\Delta/C)/T$, we have seen in remark 28 that $d_* : X \rightarrow r(X, f)$ is not a cartesian arrow for sup . However, $[d_*] : X \rightarrow r(X, f)$ is a cocartesian arrow for $[sup]$. To see that, suppose that $[\xi_*] : X \rightarrow Y$ is an arrow of $[\Delta/C]$ such that $[sup]([\xi_*]) = f$. Then, ξ_* might be consider as an arrow $(X, f) \rightarrow (Y|_{[q_Y-1]}, Y([q_Y-1] \rightarrow [q_Y]))$ in $[\Delta/C]/T$, and $[\xi_*]$ factors as $[r(\xi_*)][d_*]$ (actually, $\xi_* = r(\xi_*)d_*$). To see that this factorization is unique, suppose that another one is given, and use the fact that r preserves equivalences.

It follows that $[\Delta/C] \rightarrow C$ is a precofibration, as well as $Sd(C) \rightarrow C$. Thus, theorem 32 can be proved by using corollary 9. However, $Sd(C) \rightarrow C$ is not a cofibration in general, since cocartesian arrows are not closed under composition. This is clear because a cocartesian arrow over a non-identity map must increase the degree in exactly one.

3 Application to homotopy theory

3.1 Homotopy category of \mathcal{PoSet}

Despite the homotopy theory of partially ordered sets is largely developed, we could not find a definition for the homotopy category $Ho(\mathcal{PoSet})$. We construct it here in a suitable form, compatible with the inclusions $\mathcal{PoSet} \rightarrow Top$ and $\mathcal{PoSet} \rightarrow Top$. We will use for this purpose some well known facts about A -spaces, posets and simplicial complexes.

Recall that an A -space, or *Alexandrov space*, is a topological space in which any arbitrary intersection of open subsets is open. A topological space satisfies the T_0 separability axiom

if given two points on it, there exists an open subset that contains exactly one of these points. A T_0A -space is simply an A -space which satisfies the T_0 axiom.

There is a well-known correspondence between T_0A -spaces and preorders. We recall it briefly.

If P is a poset, let $a(P)$ be the topological space with points the elements of P and with open basis formed by the subsets $\{y | y \leq x\}$, $x \in P$. Clearly, $a(P)$ is a T_0A -space.

If X is a T_0A -space, let $s(X)$ be the poset with elements the points of X and with the order $x \leq y \iff y \in cl(x)$, where $cl(x)$ denotes the closure of $\{x\}$ in X . Note that the relation \leq is antisymmetric because X is T_0 .

35 Lemma. *The constructions $P \mapsto a(P)$ and $X \mapsto s(X)$ are functorial, and they define an equivalence of categories between \mathcal{PoSet} and the full subcategory of \mathcal{Top} whose objects are the T_0A -spaces.*

We recall some constructions from [10]. Given X a T_0A -space, a simplicial complex $k(X)$ is constructed with vertices the points of X and simplices the finite chains of $s(X)$, namely the sequences of points (x_0, \dots, x_q) satisfying $x_{i+1} \in cl(x_i)$. The construction $X \mapsto k(X)$ is functorial. Moreover, there is a natural continuous map $f_X : |k(X)| \rightarrow X$ defined by $f_X(u) = \min(\text{carrier}(u))$, where $\text{carrier}(u)$ is the unique open simplex containing u .

Given a simplicial complex K , denote by $S(K)$ its set of simplices ordered by inclusion. Define $x(K)$ as the T_0A -space associated to its simplices, namely $x(K) = aS(K)$. The construction $K \mapsto x(K)$ is functorial, since a and S are so. Moreover, since $k(x(K))$ is just the barycentric subdivision of K , there is a natural continuous map $f_K : |K| \rightarrow x(K)$ defined as the composition of the canonical homeomorphism $|K| \xrightarrow{\sim} |kx(K)|$ with the map $f_{x(K)}$. The following results are due to McCord [10].

36 Proposition. *For every T_0A -space X the map $f_X : |k(X)| \rightarrow X$ is a weak homotopy equivalence.*

37 Proposition. *For every simplicial complex K the map $f_K : |K| \rightarrow x(K)$ is a weak homotopy equivalence.*

Now we are in condition to describe $Ho(\mathcal{PoSet})$. Recall that $j : \mathcal{PoSet} \rightarrow \mathcal{Cat}$ is the functor which assigns to each poset P a category $j(P)$ in the usual way. The functor j admits a left adjoint $p : \mathcal{Cat} \rightarrow \mathcal{PoSet}$, which assigns to each small category C the poset associated to the preorder defined over the objects of C by the rule

$$X \leq Y \iff \text{there exists an arrow } X \rightarrow Y.$$

The functors $a : \mathcal{PoSet} \rightarrow \mathcal{Top}$ and $j : \mathcal{PoSet} \rightarrow \mathcal{Cat}$ embed \mathcal{PoSet} as a full reflective subcategory of \mathcal{Top} and \mathcal{Cat} . Thus, \mathcal{PoSet} inherits two definitions for weak equivalences by lifting those of \mathcal{Top} and \mathcal{Cat} . Let W_a be the class of maps $f : P \rightarrow Q$ in \mathcal{PoSet} such that $a(f) : a(P) \rightarrow a(Q)$ is a weak equivalence in \mathcal{Top} , and let W_j be the class of maps $f : P \rightarrow Q$ in \mathcal{PoSet} such that $j(f) : j(P) \rightarrow j(Q)$ is a weak equivalence in \mathcal{Cat} or, what is the same, $Bj(f) : Bj(P) \rightarrow Bj(Q)$ is a weak equivalence in \mathcal{Top} .

38 Proposition. *The classes W_a and W_j coincide.*

Proof. For each poset P there is a natural homeomorphism $Bj(P) \cong |ka(P)|$ between the classifying space of $j(P)$ and the geometric realization of McCord's construction on $a(P)$. Given $f : P \rightarrow Q$ a map in \mathcal{PoSet} , consider the following commutative diagram.

$$\begin{array}{ccccc} Bj(P) & \xrightarrow{\sim} & |ka(P)| & \xrightarrow{f_{a(P)}} & a(P) \\ Bj(f) \downarrow & & |Ka(f)| \downarrow & & a(f) \downarrow \\ Bj(Q) & \xrightarrow{\sim} & |ka(Q)| & \xrightarrow{f_{a(Q)}} & a(Q) \end{array}$$

Since the maps $f_{a(P)}$ and $f_{a(Q)}$ are weak equivalences in \mathcal{Top} (cf. proposition 36), the continuous map $Bj(f)$ is a weak equivalence if and only if $a(f)$ is so. \square

39 Definition. We say that a map $f : P \rightarrow Q$ in \mathcal{PoSet} is a *weak equivalence* if $f \in W_a = W_j$. We define the *homotopy category of \mathcal{PoSet}* , denoted $Ho(\mathcal{PoSet})$, as the localization of \mathcal{PoSet} by the family of weak equivalences.

40 Remark. It is clear that $pj(P) = P$. Unfortunately, the composition jp does not preserve homotopy types – for instance, a group G is mapped by jp into the one-arrow category. Similarly, while the composition sa is the identity functor over \mathcal{PoSet} , the other composition as fails at the homotopy level – for instance, a Hausdorff space X is mapped by as into a discrete space.

Despite last remark, the functors $j : \mathcal{PoSet} \rightarrow \mathcal{Cat}$ and $a : \mathcal{PoSet} \rightarrow \mathcal{Top}$ induce equivalences between the homotopy categories. In the next subsection we will construct homotopy inverses to the inclusions a and j .

3.2 Categorical description of $Ho(\mathcal{Top})$

The functors a and j preserve weak equivalences. Hence, they induce functors $Ho(a)$ and $Ho(j)$ at the homotopy level.

$$\begin{array}{ccccc} \mathcal{Cat} & \xleftarrow{j} & \mathcal{PoSet} & \xrightarrow{a} & \mathcal{Top} \\ \downarrow & & \downarrow & & \downarrow \\ Ho(\mathcal{Cat}) & \xleftarrow{Ho(j)} & Ho(\mathcal{PoSet}) & \xrightarrow{Ho(a)} & Ho(\mathcal{Top}) \end{array}$$

41 Theorem. *The functors $Ho(a)$ and $Ho(j)$ are equivalences of categories. Hence, the categories $Ho(\mathcal{Cat})$ and $Ho(\mathcal{Top})$ are equivalent.*

This theorem is intimately related with Quillen's theorem asserting that N induces an equivalence of categories at the homotopy level (cf. [6]). One can derive one from the other by using the well known equivalence $Ho(\mathcal{Top}) \cong Ho(s\mathcal{Set})$.

Proof. We prove first that $Ho(j)$ is an equivalence of categories. Recall from remark 24 the definition of $l : \mathcal{Cat} \rightarrow \mathcal{PoSet}$. We have seen in corollary 33 that Sd preserves weak equivalences. Since $jl = Sd^2$, it is clear that l preserves them too, hence it induces a

functor $Ho(l) : Ho(Cat) \rightarrow Ho(\mathcal{PoSet})$. We assert that l is a homotopy inverse to j , so we have to prove that there are natural isomorphisms $Ho(jl) = Ho(j)Ho(l) \cong id_{Ho(Cat)}$ and $Ho(lj) = Ho(l)Ho(j) \cong id_{Ho(\mathcal{PoSet})}$.

If we show that there are natural transformations $jl \Rightarrow id_{Cat}$ and $lj \Rightarrow id_{\mathcal{PoSet}}$ which assign to any object a weak equivalence, then by composing with the projections we will obtain natural isomorphisms, which yield another ones $Ho(jl) \cong id_{Ho(Cat)}$ and $Ho(lj) \cong id_{Ho(\mathcal{PoSet})}$ by lemma 1.

For every category C the composition $\varepsilon_C \varepsilon_{Sd(C)} : jl(C) = Sd^2(C) \rightarrow C$ is a weak equivalence by theorem 32, and clearly it is natural. This gives the natural isomorphism $Ho(jl) \cong id_{Ho(Cat)}$. The other natural isomorphism can be obtained as a restriction of this.

Now we prove that $Ho(a)$ is an equivalence of categories. We will construct an inverse to a by considering for each topological space X a simplicial complex K_X and a weak equivalence $|K_X| \rightarrow X$, which can be done naturally. We define a functor $b : Top \rightarrow \mathcal{PoSet}$ by $b(X) = S(K_X)$ the poset of simplices of the associated complex. To see that b preserves weak equivalences it is sufficient to consider the diagram

$$\begin{array}{ccccc} ab(X) = aS(K_X) & \xleftarrow{f_{K_X}} & |K_X| & \xrightarrow{\sim} & X \\ \downarrow & & \downarrow & & \downarrow \\ ab(Y) = aS(K_Y) & \xleftarrow{f_{K_Y}} & |K_Y| & \xrightarrow{\sim} & Y \end{array}$$

where f_{K_X} and f_{K_Y} are McCord's weak equivalences of proposition 37. Hence b induces a functor $Ho(b) : Ho(Top) \rightarrow Ho(\mathcal{PoSet})$.

By the same argument used above, the natural weak equivalences $|K_X| \rightarrow X$ and $|K_X| \rightarrow ab(X)$ yield natural isomorphisms at the homotopy level, which compose to give $Ho(a)Ho(b) = Ho(ab) \cong id_{Ho(Top)}$. The natural isomorphism $Ho(b)Ho(a) = Ho(ba) \cong id_{Ho(\mathcal{PoSet})}$ can be obtained as a restriction of the previous one. \square

42 Remark. By the work of Thomason [15] we know that Cat admits a closed model structure, weak equivalences being the ones we work with. By the corrections made by Cisinski [2] over the paper of Thomason, we know that every cofibrant category under this structure is a poset. Thus, the equivalence $Ho(\mathcal{PoSet}) \xrightarrow{\sim} Ho(Cat)$ can be deduced from the composition

$$Ho(Cat_c) \xrightarrow{\sim} Ho(\mathcal{PoSet}) \xrightarrow{\sim} Ho(Cat),$$

where Cat_c denotes the full subcategory of cofibrant objects.

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